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Dirac reduction of dual Poisson–presymplectic pairs

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Abstract

A new notion of a dual Poisson–presymplectic pair is introduced and its properties are examined. The procedure of Dirac reduction of Poisson operators onto submanifolds proposed by Dirac is in this paper embedded in a geometric procedure of reduction of dual Poisson–presymplectic pairs. The method presented generalizes those introduced by J Marsden and T Ratiu for reductions of Poisson manifolds. Two examples are given.

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1. Introduction

In [1] Dirac introduced a method of reducing a given Poisson bracket onto a submanifold $\mathcal{S}$ given by some constraints φ. The geometric meaning of this reduction procedure has been investigated in [2] and also in [3]. In this paper we complete this picture by its 'dual' part by developing a theory of Marsden–Ratiu-type reduction of presymplectic 2-forms Ω that are (in a sense developed below) dual to a given Poisson operator Π.

This paper is organized as follows. In this section we recall some basic notions from Poisson and presymplectic geometry. In section 2 we introduce and discuss the novel notion of a dual Poisson–presymplectic (dual P–p) pair. We also examine some basic properties of P–p pairs. In section 3 we present a geometric reduction procedure of such a pair to any submanifold that its tangent bundle contains the kernel of the presymplectic form that enters our P–p pair. This is the main section of this paper. We conclude this paper by section 4 containing two examples.

Given a manifold $\mathcal{M}$, a *Poisson operator* Π on $\mathcal{M}$ is a bivector, $\Pi \in \Lambda^2(\mathcal{M})$ (degenerate in general), such that its Schouten bracket with itself vanishes. A function $c : \mathcal{M} \to \mathbf{R}$ is called a *Casimir function* of the Poisson operator Π if for any function $f : \mathcal{M} \to \mathbf{R}$ we have

$\{f, c\}_\Pi = 0$ (or, equivalently, if $\Pi\,\mathrm{d}c = 0$). A vector field X_f related to a function f by the relation

$$X_f = \Pi\,\mathrm{d}f \tag{1}$$

is called a Hamiltonian vector field with respect to the Poisson operator Π. If X is any vector field on $\mathcal{M}$ that is Hamiltonian with respect to Π then $L_X\Pi = 0$, where L_X is the Lie-derivative operator in the direction X.

Further, a *presymplectic form* Ω on $\mathcal{M}$ is a 2-form that is closed (degenerate in general). The kernel of any presymplectic form of constant rank is always integrable. A vector field X_f related to a function f by the relation

$$\Omega X_f = \mathrm{d}f \tag{2}$$

is called an inverse Hamiltonian vector field with respect to the presymplectic operator Ω. For a closed 2-form Ω if $\Omega(Y) = 0$ for some vector field Y on $\mathcal{M}$ then $L_Y\Omega = 0$.

Note that when Π is nondegenerate one can always define $\Omega = \Pi^{-1}$, and then equations (1) and (2) are equivalent and a vector field that is Hamiltonian with respect to Π is simultaneously inverse Hamiltonian with respect to Ω. In the degenerate case we encounter problems. Firstly, one cannot define Ω as the inverse of Π. Secondly, for a degenerate Π equation (1) defines a Hamiltonian vector field for any function f (as in the nondegenerate case), while for a degenerate Ω and arbitrary f there is usually no vector field X_f such that (2) is fulfilled. In other words, equation (2) is valid only for a particular class of functions (contrary to the nondegenerate case). We will try to overcome these difficulties in the next section. We will constantly assume that our degenerate operators are of constant rank.

2. Dual Poisson–presymplectic pairs

In this section we introduce and discuss the notion of a dual Poisson–presymplectic pair.

Consider a smooth manifold $\mathcal{M}$ of dimension m equipped with a pair of antisymmetric operators Π, Ω.

Definition 1. *A pair of antisymmetric tensor fields* (Π, Ω) *such that* $\Pi : T^*\mathcal{M} \to T\mathcal{M}$ *(i.e.* Π *is twice contravariant) and* $\Omega : T\mathcal{M} \to T^*\mathcal{M}$ *(i.e.* Ω *is twice covariant) is called a dual pair if there exists* $r, 0 \leqslant r \leqslant m$*, functionally independent scalar functions* $c_i : \mathcal{M} \to \mathbf{R}, i = 1, \ldots, r$*, and* r *linearly independent vector fields* $Y_i, i = 1, \ldots, r$*, such that the following conditions are satisfied:*

1. $Y_i(c_j) = \delta_{ij}$ *for all* $i, j = 1, \ldots, r$.
2. *The kernel of* Π *is spanned by the differentials* $\mathrm{d}c_i$, $\ker(\Pi) = Sp\{\mathrm{d}c_i\}_{i=1,\ldots,r}$.
3. *The kernel of* Ω *is spanned by the vector fields* Y_i, $\ker(\Omega) = Sp\{Y_i\}_{i=1,\ldots,r}$.
4. *The following partition of identity holds on* $T\mathcal{M}$,

$$I = \Pi\Omega + \sum_{i=1}^{r} Y_i \otimes \mathrm{d}c_i \tag{3}$$

where $\otimes$ *denotes the tensor product.*

We would like to point out that our definition of a dual pair means something different from the definition of a dual pair introduced by Weinstein in [6]. Besides, we mostly work with dual pairs consisting of Poisson operators and closed forms (which we call 'dual P–p pairs', see below), so hopefully it will not cause any confusion.

The partition of identity (3) reads on $T^*\mathcal{M}$

$$I = \Omega^*\Pi^* + \sum_{i=1}^{r} \mathrm{d}c_i \otimes Y_i$$

which due to antisymmetry of Π and Ω yields

$$I = \Omega\Pi + \sum_{i=1}^{r} \mathrm{d}c_i \otimes Y_i. \tag{4}$$

Let us denote the foliation of $\mathcal{M}$ associated with the functions c_i by $\mathcal{N}$. This foliation consists of level submanifolds $\mathcal{N}_\nu$ of functions c_i, $\mathcal{N}_\nu = \{x \in M : c_i(x) = \nu_i, i = 1, \ldots, r\}, \nu = (\nu_r, \ldots, \nu_r)$. Condition 1 of definition 1 implies that the distribution $\mathcal{Y}$ spanned by the vector fields Y_i is transversal to the foliation $\mathcal{N}$, i.e. no vector in $\mathcal{Y}$ is ever tangent to the foliation $\mathcal{N}$. Thus, for any $x \in \mathcal{M}$ we have

$$T_x\mathcal{M} = T_x\mathcal{N}_\nu \oplus Y_x \qquad T_x^*\mathcal{M} = T_x^*\mathcal{N}_\nu \oplus Y_x^*$$

where $\mathcal{N}_\nu$ is a submanifold from the foliation $\mathcal{N}$ that passes through x, the symbol $\oplus$ denotes the direct sum of the vector spaces, Y_x is the subspace of $T_x\mathcal{M}$ spanned by the vectors Y_i at this point, $T_x^*\mathcal{N}_\nu$ is the annihilator of Y_x and Y_x^* is the annihilator of $T_x\mathcal{N}_\nu$. Condition 2 of definition 1 implies that the image $\mathrm{Im}(\Pi)$ is at every point tangent to a level submanifold $\mathcal{N}_\nu$ that passes through this point. Indeed, if $\Pi\,\mathrm{d}c_i = 0$ then for any 1-form α we have, due to the antisymmetry of Π, that $\langle \mathrm{d}c_i, \Pi\alpha\rangle = -\langle \alpha, \Pi\,\mathrm{d}c_i\rangle = 0$ for all $i = 1, \ldots, r$, so that the vector $\Pi\alpha$ is always tangent to $\mathcal{N}$. Condition 3 means that $\mathrm{Im}(\Omega)$ is at every point x contained in $T_x^*\mathcal{N}_\nu$ (again for appropriate ν). Indeed, if $\Omega(Y_i) = 0$ for all $i = 1, \ldots, r$, then for any vector field V we have (due to the antisymmetry of Ω) $\langle \Omega V, Y_i\rangle = -\langle \Omega Y_i, V\rangle = 0$. Condition 4 is the most interesting one: obviously, it describes the degree of degeneracy of our pair. But if we restrict our attention to those dual pairs that consist of a Poisson operator and of a closed 2-form then it has yet another, deeper meaning.

Definition 2. *A dual pair* (Π, Ω) *is called a dual Poisson–presymplectic pair (in short, dual P–p pair) if* Π *is Poisson and if* Ω *is closed.*

Remark 3. In the case where a dual P–p pair has no degeneration ($r = 0$) we get the usual Poisson–symplectic pair of mutually inverse operators, since (3) reads then as $I = \Pi\Omega$. In the case where $r = m$ we have full degeneration: $\Pi = 0$ and $\Omega = 0$ as then $Sp\{\mathrm{d}c_i\}_{i=1,\ldots,r} = T^*\mathcal{M}$ and $Sp\{Y_i\}_{i=1,\ldots,r} = T\mathcal{M}$. This case will therefore be excluded as non-interesting.

Let (Π, Ω) be a dual P–p pair and let

$$X_f = \Pi\,\mathrm{d}f \tag{5}$$

be a Hamiltonian vector field with respect to Π. Applying Ω to both sides of (5) we get

$$\mathrm{d}f = \Omega(X_f) + \sum_{i=1}^{r} Y_i(f)\,\mathrm{d}c_i. \tag{6}$$

In that sense Ω plays the role of the 'inverse' of Π. Note that vector fields that are Hamiltonian with respect to Ω are precisely those that are related to functions f which are annihilated by $\ker(\Omega)$. For such functions (6) reduces to (2).

Proposition 4. *For a dual P–p pair* (Π, Ω) *the vector fields* Y_i *mutually commute:* $[Y_i, Y_j] = 0, i, j = 1, \ldots, r$.

Proof. Since Ω is presymplectic, $\ker(\Omega)$ is an integrable distribution so that

$$[Y_i, Y_j] = \sum_{k=1}^{r} \phi_{ij}^k Y_k$$

where ϕ_{ij}^k are some functions on $\mathcal{M}$. Evaluating this relation on all Casimirs c_l we immediately find that $\phi_{ij}^k = 0$ for all $i, j, k = 1, \ldots, r$. □

Of course, the vector fields Y_i and the forms $\mathrm{d}c_i$ in the definition of a dual pair are not unique. For example, we can change the basis in the distribution spanned by Y_i and compensate it by a change of basis in the distribution spanned by $\mathrm{d}c_i$. We have however the following uniqueness theorem:

Theorem 5 (uniqueness theorem)**.** *Suppose that* (Π, Ω) *and* (Π, Ω') *are two dual P–p pairs that share the same* c_i *and such that* $\ker(\Omega) = \ker(\Omega')$*. Then* $\Omega = \Omega'$*.*

Proof. Since $\ker(\Omega) = \ker(\Omega')$ then $Y_i' = \sum_{j=1}^{r} \lambda_{ij} Y_j$ for some functions λ_{ij} such that $\det(\lambda_{ij}) \neq 0$. Thus

$$\delta_{ij} = Y_i'(c_j) = \sum_{s=1}^{r} \lambda_{is} Y_s(c_j) = \sum_{s=1}^{r} \lambda_{is} \delta_{sj} = \lambda_{ij}$$

so that $Y_i' = Y_i$ for all i. Thus, since $I = \Pi\Omega + \sum_{i=1}^{r} Y_i \otimes \mathrm{d}c_i$ and $I = \Pi\Omega' + \sum_{i=1}^{r} Y_i \otimes \mathrm{d}c_i$ we have that $\Pi(\Omega' - \Omega) = 0$, which implies that $\Omega' - \Omega = 0$ since the product of two antisymmetric operators is zero only if (at least) one of them is zero. □

The question thus arises: what is the actual 'gauge freedom' for a given dual P–p pair? In other words, given a dual P–p pair (Π, Ω) how can we deform Ω to a new presymplectic form Ω' so that (Π, Ω') is again dual or how can we deform Π to a new Poisson operator Π' so that (Π', Ω) is also a dual P–p pair? An example of such a gauge freedom is given below.

Proposition 6. *Let* (Π, Ω) *be a dual P–p pair as in definitions 1 and 2. Define*

$$\Omega' = \Omega + \sum_i \mathrm{d}f_i \wedge \mathrm{d}c_i$$

where f_i *are some real functions on* $\mathcal{M}$*. Then* (Π, Ω') *is a dual P–p pair with* $\ker(\Omega') = Sp\left\{Y_i' = Y_i - \Pi\,\mathrm{d}f_i\right\}$ *provided that*

$$Y_i(f_j) - Y_j(f_i) + \{f_i, f_j\}_\Pi = 0 \qquad \text{for all } i, j. \tag{7}$$

The proof is by direct computation.

Before we consider a gauge freedom for the operator Π we prove a useful lemma.

Lemma 7. *Let* (Π, Ω) *be a dual P–p pair. Then*

$$L_{Y_i}\Pi = 0 \qquad i = 1, \ldots, r. \tag{8}$$

Proof. From the partition of identity and the property $L_{Y_i}\Omega = 0$ we have

$$\begin{aligned} 0 = L_{Y_i} I &= \left(L_{Y_i}\Pi\right)\Omega + \Pi\left(L_{Y_i}\Omega\right) + \sum_{j=1}^{r} [Y_i, Y_j] \otimes \mathrm{d}c_j \\ &= \left(L_{Y_i}\Pi\right)\Omega. \end{aligned}$$

On the other hand, from the property $\Pi\, \mathrm{d}c_j = 0$, it follows that

$$0 = L_{Y_i}(\Pi\, \mathrm{d}c_j) = \left(L_{Y_i}\Pi\right) \mathrm{d}c_j \qquad \text{as} \quad L_{Y_i}\, \mathrm{d}c_j = \mathrm{d}(\delta_{ij}) = 0.$$

Thus, from the decomposition (6), for any function f we have

$$\left(L_{Y_i}\Pi\right) \mathrm{d}f = \left(L_{Y_i}\Pi\right) \left(\Omega(X_f) + \sum_{i=1}^{r} Y_i(f)\, \mathrm{d}c_i\right) = 0$$

and arbitrariness of f implies that $L_{Y_i}\Pi = 0$. □

Proposition 8. *Suppose that* (Π, Ω) *is a dual P–p pair. Suppose that* K_i, $i = 1, \ldots, r$, *are vector fields that are Hamiltonian with respect to* Π *and inverse Hamiltonian with respect to* Ω, *i.e.*

$$\Omega(K_i) = \mathrm{d}H_i \qquad K_i = \Pi\, \mathrm{d}H_i \tag{9}$$

for some functions H_i *and such that*

$$\Omega(K_i, K_j) = 0 \qquad \text{for all } i, j. \tag{10}$$

Then the pair (Π', Ω) *with* $\Pi' = \Pi + \sum_{i=1}^{r} Y_i \wedge K_i$ *is a dual P–p pair with vector fields* Y_i *and Casimirs* $c'_i = c_i + H_i$.

Proof. An easy calculation with the use of partition of identity and the assumptions (9) yields

$$\Pi'\Omega = I - \sum_i Y_i \otimes \mathrm{d}c'_i$$

so that the partition of identity for (Π', Ω) is satisfied. Moreover,

$$Y_i(H_j) = \langle \Omega K_j, Y_i \rangle = -\langle \Omega Y_i, K_j \rangle = 0$$

as $\Omega(Y_i) = 0$, so that

$$Y_i(c'_j) = Y_i(c_j) + Y_i(H_j) = Y_i(c_j) = \delta_{ij}.$$

Further, $K_i(c_j) = \langle \mathrm{d}c_j, \Pi\, \mathrm{d}H_i \rangle = -\langle \mathrm{d}H_i, \Pi\, \mathrm{d}c_j \rangle = 0$, which implies that

$$\Pi'\, \mathrm{d}c'_j = \sum_i \Omega(K_j, K_i) Y_i = 0$$

due to the assumption (10).

One can also show (by using lemma 7) that the Schouten bracket $[\Pi', \Pi']_S$ vanishes so that Π' is indeed Poisson. □

Let us now turn our attention to brackets induced on the space $C^\infty(\mathcal{M})$ of all smooth real-valued functions on $\mathcal{M}$.

We know that the Poisson operator Π turns the space $C^\infty(\mathcal{M})$ of all smooth real-valued functions on $\mathcal{M}$ into a Poisson algebra with the Poisson bracket

$$\{F, G\}_\Pi = \langle \mathrm{d}F, \Pi\, \mathrm{d}G \rangle. \tag{11}$$

In the case where Ω is a part of a dual P–p pair we can define the above bracket through the action of Ω on X_F and X_G.

Proposition 9. *Let* (Π, Ω) *be a dual P–p pair. Define a new bracket on* $C^\infty(\mathcal{M})$ *through*

$$\{F, G\}^\Omega = \Omega(X_F, X_G)$$

where as usual $X_F = \Pi\, \mathrm{d}F$ *and* $X_G = \Pi\, \mathrm{d}G$. *Then* $\{\cdot\,, \cdot\}^\Omega = \{\cdot\,, \cdot\}_\Pi$, *i.e. both brackets are identical.*

Proof. A simple calculation yields

$$\Omega(X_F, X_G) = \langle \Omega X_F, X_G\rangle = \langle \Omega\Pi\, \mathrm{d}F, \Pi\, \mathrm{d}G\rangle \overset{*}{=} \\ = \langle \mathrm{d}F, \Pi\, \mathrm{d}G\rangle - \sum_i Y_i(F)\langle \mathrm{d}c_i, \Pi\, \mathrm{d}G\rangle = \langle \mathrm{d}F, \Pi\, \mathrm{d}G\rangle$$

as $\langle \mathrm{d}c_i, \Pi\, \mathrm{d}G\rangle = -\langle \mathrm{d}G, \Pi\, \mathrm{d}c_i\rangle = 0$. The equality with the asterisk is due to the partition of identity on $T^*\mathcal{M}$ (4). □

Let us now present two examples of P–p pairs.

Example 10. Consider a manifold $\mathcal{M}$ parametrized locally by coordinates

$$(q_1, \ldots, q_n, p_1, \ldots, p_n, c_1, \ldots, c_r)$$

and a pair of operators that in these coordinates have the form

$$\Pi = \left[\begin{array}{cc|c} 0_n & I_n & \\ -I_n & 0_n & 0_{2n\times r} \\ \hline \multicolumn{2}{c|}{0_{r\times 2n}} & 0_r \end{array}\right] \qquad \Omega = \left[\begin{array}{cc|c} 0_n & -I_n & \\ I_n & 0_n & 0_{2n\times r} \\ \hline \multicolumn{2}{c|}{0_{r\times 2n}} & 0_r \end{array}\right] \tag{12}$$

i.e. Π is the canonical Poisson operator with r Casimirs c_i while Ω is the canonical presymplectic form with the kernel spanned by $Y_i = \frac{\partial}{\partial c_i}$. Here and in what follows I_n denotes an $n \times n$ identity matrix and in general subscripts by a matrix block denote the dimensions of this block. Obviously, $Y_i(c_j) = \delta_{ij}$. Moreover, the product $\Pi\Omega$ has the form

$$\Pi\Omega = \left[\begin{array}{c|c} I_{2n} & 0_{2n\times r} \\ \hline 0_{r\times 2n} & 0_r \end{array}\right]$$

while the tensor product $\sum_{i=1}^{r} Y_i \otimes \mathrm{d}c_i$ has the form

$$\sum_{i=1}^{r} Y_i \otimes \mathrm{d}c_i = \left[\begin{array}{c|c} 0_{2n} & 0_{2n\times r} \\ \hline 0_{r\times 2n} & I_r \end{array}\right]$$

and thus the partition of identity (3) is satisfied. Thus, all the conditions of definition 1 are satisfied. Note also that both Π and Ω have constant rank $2n$. This simple example will be further developed in section 4.

We will now present a non-canonical example.

Example 11. In our second example we consider a five-dimensional manifold $\mathcal{M}$ parametrized (locally) by coordinates (q_1, q_2, p_1, p_2, e) and a pair of operators given by

$$\Pi = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2}q_1 & p_1 \\ 0 & 0 & \frac{1}{2}q_1 & q_2 & p_2 \\ 0 & -\frac{1}{2}q_1 & 0 & -\frac{1}{2}p_1 & 0 \\ -\frac{1}{2}q_1 & -q_2 & \frac{1}{2}p_1 & 0 & e \\ -p_1 & -p_2 & 0 & -e & 0 \end{bmatrix} \tag{13}$$

and

$$\Omega = \begin{bmatrix} 0 & 2\frac{p_1}{q_1^2} & 4\frac{q_2}{q_1^2} & -2\frac{1}{q_1} & \frac{1}{2}(p_1p_2 - q_1e) \\ -2\frac{p_1}{q_1^2} & 0 & -2\frac{1}{q_1} & 0 & -\frac{1}{2}p_1^2 \\ -4\frac{q_2}{q_1^2} & 2\frac{1}{q_1} & 0 & 0 & \frac{1}{2}q_1p_2 - q_2p_1 \\ 2\frac{1}{q_1} & 0 & 0 & 0 & \frac{1}{2}q_1p_1 \\ \frac{1}{2}(q_1e - p_1p_2) & \frac{1}{2}p_1^2 & q_2p_1 - \frac{1}{2}q_1p_2 & -\frac{1}{2}q_1p_1 & 0 \end{bmatrix} \tag{14}$$

Both these operators have constant rank 4 (generically). It is easy to check that Π is Poisson and that Ω is closed. The only (independent) Casimir of Π is given by

$$c = -\tfrac{1}{2}q_2 p_1^2 + \tfrac{1}{2}q_1 p_1 p_2 - \tfrac{1}{4}e q_1^2 \qquad (15)$$

while the kernel of Ω is spanned by the vector field

$$Y = p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2} + e \frac{\partial}{\partial p_2} - \frac{4}{q_1^2}\frac{\partial}{\partial e}.$$

Naturally, $Y(c) = 1$. The explicit form of the tensor $Y \otimes \mathrm{d}c$ is rather complicated but a direct calculation shows that $\Pi\Omega + Y \otimes \mathrm{d}c = I$. Thus, (Π, Ω) is a dual P–p pair. This example will also be developed later on.

The first example (example 10) illustrates that the following existence statement must be true.

Proposition 12. *For a given Poisson operator* Π *of constant rank on* $\mathcal{M}$*, there always exists (locally) a closed 2-form* Ω *such that* (Π, Ω) *is a dual P–p pair and vice versa; for a given presymplectic form* Ω *of constant rank, there always exists a Poisson operator* Π *such that* (Π, Ω) *is a dual P–p pair.*

Proof. Given a Poisson operator Π (a closed 2-form Ω) it is enough to pass to the Darboux coordinates for Π (Ω) and choose Ω (Π) as in example 10. □

3. Dirac reduction of Poisson–presymplectic pairs

Consider now a smooth m-dimensional manifold $\mathcal{M}$ endowed with a dual P–p pair (Π, Ω) as in definitions 1 and 2 and a smooth s-dimensional foliation $\mathcal{S}$ of $\mathcal{M}$ with the leaves $\mathcal{S}_\nu = \{x \in \mathcal{M} : \varphi_i(x) = \nu_i, \nu_i \in \mathbf{R}, i = 1, \ldots, k\}$ given by s functionally independent functions $\varphi_i : M \to \mathbf{R}$. In this section we present a procedure of reducing the dual P–p pair (Π, Ω) to a dual P–p pair (π_R, ω_R) on any leaf $\mathcal{S}_\nu$ of $\mathcal{S}$ provided that some additional assumptions about the relative positions of the foliations $\mathcal{N}$ and $\mathcal{S}$ hold and that we are in a generic case that will be called the Dirac case. This reduction will be similar to the ideas developed by Marsden and Ratiu (see [2, 3]).

Let us thus fix a distribution $\mathcal{Z}$ (to be determined later) of constant dimension $k = m - s$ (that is a smooth collection of k-dimensional subspaces $\mathcal{Z}_x \subset T_x\mathcal{M}$ at every point x in $\mathcal{M}$) that is transversal to $\mathcal{S}$ in the sense that no vector field $Z \in \mathcal{Z}$ is at any point tangent to the foliation $\mathcal{S}$. Hence we have

$$T_x\mathcal{M} = T_x\mathcal{S}_\nu \oplus \mathcal{Z}_x$$

for every $x \in \mathcal{S}_\nu$, and similarly

$$T_x^*\mathcal{M} = T_x^*\mathcal{S}_\nu \oplus \mathcal{Z}_x^*$$

where $T_x^*\mathcal{S}$ is the annihilator of $\mathcal{Z}_x$ and $\mathcal{Z}_x^*$ is the annihilator of $T_x\mathcal{S}$. These distributions are assumed to be regular, i.e. there exist linearly independent vector fields Z_i, $i = 1, \ldots, k$, such that $\mathcal{Z} = Sp\{Z_i\}_{i=1,\ldots,k}$. Without loss of generality we can assume that the vector fields Z_i are chosen so that the following normalization condition holds:

$$Z_i(\varphi_j) = \delta_{ij}.$$

There exists a natural projection $X_\parallel$ of an arbitrary vector field X on $\mathcal{M}$ along $\mathcal{Z}$ onto the foliation $\mathcal{S}$ given by

$$X_\parallel = X - \sum_{i=1}^{k} X(\varphi_i) Z_i$$

as obviously $X_{\|}(\varphi_i) = 0$. Similarly, any 1-form α can be naturally projected along $\mathcal{Z}$ to a 1-form $a_{\|}$ on $T^*\mathcal{S}$ as follows,

$$\alpha_{\|} = \alpha - \sum_{i=1}^{k} \alpha(Z_i)\,\mathrm{d}\varphi_i \tag{16}$$

since $\alpha_{\|}(Z_i) = 0$. Finally, let us define the vector fields X_i, $i = 1, \ldots, s$, as the Hamiltonian vector fields

$$X_i = \Pi\,\mathrm{d}\varphi_i.$$

Definition 13. *A function $F : \mathcal{M} \to \mathbf{R}$ is invariant with respect to $\mathcal{Z}$ if $L_Z F = Z(F) = 0$ for any $Z \in \mathcal{Z}$.*

Definition 14. *A Poisson operator Π is invariant with respect to the distribution $\mathcal{Z}$ if $\{F, G\}_\Pi$ is a $\mathcal{Z}$-invariant function for any pair of $\mathcal{Z}$-invariant functions F and G. That is, if $L_{Z_i} F = L_{Z_i} G = 0$ then $L_{Z_i}\{F, G\}_\Pi = 0$.*

Let us now consider a $\mathcal{Z}$-invariant Poisson operator Π and define the following bilinear map:

$$\Pi_D(\alpha, \beta) = \Pi(\alpha_{\|}, \beta_{\|}) \text{ for any pair } \alpha, \beta \text{ of 1-forms.} \tag{17}$$

This new mapping induces a new bracket for functions on $\mathcal{M}$,

$$\{F, G\}_{\Pi_D} = \Pi_D(\mathrm{d}F, \mathrm{d}G) = \Pi((\mathrm{d}F)_{\|}, (\mathrm{d}G)_{\|})$$

and thus it is easy to show that the corresponding bivector Π_D has the following form,

$$\Pi_D = \Pi - \sum_i X_i \wedge Z_i + \frac{1}{2}\sum_{i,j} \varphi_{ij} Z_i \wedge Z_j \tag{18}$$

which we can treat as a deformation of the original Poisson bivector Π. Here the functions φ_{ij} are defined as

$$\varphi_{ij} = \{\varphi_i, \varphi_j\}_\Pi = X_j(\varphi_i). \tag{19}$$

Theorem 15. *For any $x \in \mathcal{M}$*

$$\Pi_D(\alpha_x) \in T_x\mathcal{S} \qquad \text{for any} \quad \alpha_x \in T_x^*\mathcal{M}$$

i.e. the image of Π_D is tangent to the foliation $\mathcal{S}$.

Proof. We have to show that $\Pi_D(\mathrm{d}\varphi_k) = 0$ for all k. According to (18) we have

$$\begin{aligned}\Pi_D(\mathrm{d}\varphi_k) &= X_k - \sum_i \delta_{ik} X_i + \sum_i \varphi_{ki} Z_i + \frac{1}{2}\sum_{i,j} \varphi_{ij}(\delta_{jk} Z_i - \delta_{ik} Z_j) \\ &= \sum_i \varphi_{ki} Z_i + \frac{1}{2}\sum_i \varphi_{ik} Z_i - \frac{1}{2}\sum_j \varphi_{kj} Z_j = 0\end{aligned}$$

due to skewsymmetry of φ_{ij}. □

Theorem 16. *If a Poisson operator Π is $\mathcal{Z}$-invariant then the bivector (18) is Poisson.*

Proof. The operator Π_D is obviously antisymmetric and the corresponding bracket $\{\cdot, \cdot\}_{\Pi_D}$ satisfies the Leibniz rule. The Jacobi identity for $\{\cdot\,, \cdot\}_{\Pi_D}$

$$\left\{\{F, G\}_{\Pi_D}, H\right\}_{\Pi_D} + \text{cycl.} = 0$$

reads due to (17)

$$\langle (\mathrm{d}\langle (\mathrm{d}F)_\parallel, \Pi(\mathrm{d}G)_\parallel\rangle)_\parallel, \Pi(\mathrm{d}H)_\parallel\rangle + \text{cycl.} = 0. \tag{20}$$

But due to (16)

$$(\mathrm{d}\langle (\mathrm{d}F)_\parallel, \Pi(\mathrm{d}G)_\parallel\rangle)_\parallel = \mathrm{d}\langle (\mathrm{d}F)_\parallel, \Pi(\mathrm{d}G)_\parallel\rangle - \sum_i Z_i(\langle (\mathrm{d}F)_\parallel, \Pi(\mathrm{d}G)_\parallel\rangle)\,\mathrm{d}\varphi_i$$

and at every point $x \in \mathcal{M}$ we have $(\mathrm{d}F)_\parallel, (\mathrm{d}G)_\parallel \in T^*\mathcal{S}$ so that

$$Z_i(\langle (\mathrm{d}F)_\parallel, \Pi(\mathrm{d}G)_\parallel\rangle) = \left\langle (\mathrm{d}F)_\parallel, \left(L_{Z_i}\Pi\right)(\mathrm{d}G)_\parallel\right\rangle = 0.$$

The last equality is fulfilled due to the assumed $\mathcal{Z}$-invariance of Π. Thus, the Jacobi identity (20) actually reads

$$\langle \mathrm{d}\langle (\mathrm{d}F)_\parallel, \Pi(\mathrm{d}G)_\parallel\rangle, \Pi(\mathrm{d}H)_\parallel\rangle + \text{cycl.} = 0$$

and is obviously satisfied due to the Jacobi identity for Π. □

Since the deformed operator Π_D is Poisson and since the functions φ_i are its Casimirs, we can properly restrict Π_D to any of its symplectic leaves $\mathcal{S}_\nu$ obtaining a reduced Poisson operator π_R on every leaf $\mathcal{S}_\nu$:

$$\pi_R \stackrel{\text{def}}{=} \Pi_D|_{\mathcal{S}_\nu}.$$

Up to now the distribution $\mathcal{Z}$ was not fully determined. In the generic case (that we call the *Dirac case*), when all the vector fields X_i are transversal to the foliation $\mathcal{S}$, we can choose the distribution $\mathcal{Z}$ simply as the span of the vector fields X_i, $\mathcal{Z} = Sp\{X_i\}_{i=1,\ldots,k}$. We can now define our vector fields Z_i as a new basis of $\mathcal{Z}$:

$$Z_i = \sum_{j=1}^{k} (\varphi^{-1})_{ji} X_j \qquad i = 1, \ldots, k. \tag{21}$$

Indeed, since $\det(\varphi) \neq 0$ the vector fields Z_i also span the distribution $\mathcal{Z}$ and satisfy the normalization condition, $\langle \mathrm{d}\varphi_i, Z_j\rangle = Z_j(\varphi_i) = \delta_{ij}$, as

$$Z_j(\varphi_i) = \sum_{s=1}^{k} (\varphi^{-1})_{sj} X_s(\varphi_i) = \sum_{s=1}^{k} (\varphi^{-1})_{sj} \varphi_{is} = \delta_{ij}.$$

Moreover, such choice of Z_i makes the operator Π $\mathcal{Z}$-invariant: if $L_{X_i}F = L_{X_i}G = 0$ for all i then $L_{X_i}\{F, G\}_\Pi = \left\langle \mathrm{d}F, \left(L_{X_i}\Pi\right)\mathrm{d}G\right\rangle = 0$ since $L_{X_i}\Pi = 0$ ($\{X_i\}$ is just another basis of $\mathcal{Z}$). In this case the deformation (18) attains the form

$$\Pi_D = \Pi - \frac{1}{2}\sum_{i=1}^{k} X_i \wedge Z_i \tag{22}$$

and is, as mentioned above, Poisson. It is easy to check that this operator defines the following bracket on $C^\infty(\mathcal{M})$,

$$\{F, G\}_{\Pi_D} = \{F, G\}_\Pi - \sum_{i,j=1}^{k} \{F, \varphi_i\}_\Pi (\varphi^{-1})_{ij} \{\varphi_j, G\}_\Pi \tag{23}$$

(where $F, G : \mathcal{M} \to \mathbf{R}$ are two *arbitrary* functions on $\mathcal{M}$) which is just the well-known Dirac deformation [1] of the bracket $\{.,.\}_\Pi$ associated with Π. As we will show below in a more general context, $\ker(\Pi_D) = Sp\{\mathrm{d}\varphi_i, \mathrm{d}c_j\}_{i=1,\ldots,k,\, j=1,\ldots,r}$, i.e. the Dirac deformation preserves all the old Casimir functions c_i and introduces new Casimirs φ_i.

In the case when all the vector fields X_i are tangent to the foliation $\mathcal{S}$ (we call this case the *tangent case*) the foliation $\mathcal{S}$ is Lagrangian with respect to any Ω dual to Π. Then the deformation (18) attains the form

$$\Pi_D = \Pi - \sum_{i=1}^{k} X_i \wedge Z_i \tag{24}$$

and has been considered in [7, 8].

Let us observe that formula (18) can be rewritten as $\Pi_D = \Pi - \sum_i V_i \wedge Z_i$ with $V_i = X_i - \frac{1}{2}\sum_j \varphi_{ji} Z_j$. A generalization of this formula has been considered in [3] where the vector fields V_i were determined only up to a functional equation $V_i = X_i + \sum_j V_j(\varphi_i) Z_j$. Two natural solutions to this equation in the above-mentioned two cases are $V_i = \frac{1}{2} X_i$ (in the Dirac case) or $V_i = X_i$ (in the tangent case) and they yield exactly the two above deformations (22) and (24) respectively.

In any case, our process of reducing the operator Π to π_R consists of two steps: we first deform Π to Π_D and then reduce in a natural way Π_D to π_R through a plain restriction: $\pi_R = \Pi_D|_{\mathcal{S}}$.

Our construction generalizes the construction of Marsden and Ratiu in the following sense. Marsden and Ratiu presented in [2] a natural way of reducing a given Poisson bracket $\{\cdot,\cdot\}_\Pi$ on $\mathcal{M}$ to a Poisson bracket $\{\cdot,\cdot\}_{\pi_R}$ on a given submanifold $\mathcal{S}_0$ (in our notation). Their method is non-constructive in the sense that in order to find the bracket $\{f,g\}_{\pi_R}$ of two functions $f, g : S_0 \to R$ one has to calculate $\mathcal{Z}|_{\mathcal{S}_0}$-invariant prolongations of these functions. Our construction is performed on the level of bivectors rather than on the level of Poisson brackets. This construction (by deformation of the bivector Π) applies directly to every leaf of the distribution $\mathcal{S}$ and moreover it is constructive. At every leaf, however, both constructions are equivalent. Also, our construction can be extended to a similar construction for closed 2-forms, as shown below. On the other hand, we make the assumption about the transversality of the distribution $\mathcal{Z}$ that was not present in the original paper of Marsden and Ratiu. This assumption is, however, very natural since it makes all the assumptions of the Poisson reduction theorem in [2] automatically satisfied.

Now we turn to an analogous question of reducing closed 2-forms onto the foliation $\mathcal{S}$. Of course, there always exists a natural restriction of any closed 2-form on any submanifold $\mathcal{S}_\nu$, obtained simply by restricting its domain to $T\mathcal{S}_\nu$. However, in the case that our closed 2-form is a part of a dual P–p pair (Π, Ω) it is also natural to consider a similar two-step procedure, where we first deform Ω to Ω_D (such that (Π_D, Ω_D) is again a dual pair) and then restrict Ω_D to a closed 2-form ω_R on $\mathcal{S}_\nu$ such that (π_R, ω_R) is a dual P–p pair. This is the main aim of this paper.

Let us thus define, in analogy with (17), the following bilinear map:

$$\Omega_D(U, V) = \Omega(U_\parallel, V_\parallel) \text{ for any vector fields } U, V \text{ on } \mathcal{M}. \tag{25}$$

This map induces the following 2-form Ω_D on $\mathcal{M}$:

$$\Omega_D = \Omega - \sum_{i=1}^{k} \xi_i \wedge \mathrm{d}\varphi_i - \frac{1}{2}\sum_{i,j=1}^{k} \xi_i(Z_j)\, \mathrm{d}\varphi_i \wedge \mathrm{d}\varphi_j \tag{26}$$

where the 1-forms ξ_i are defined as

$$\xi_i = \Omega(Z_i).$$

The 2-form Ω_D obviously restricts to the same 2-form on $\mathcal{S}_\nu$ as Ω does. That is, we can define a form ω_R on every leaf of $\mathcal{S}$ through the plain restriction of Ω_D (or Ω) to $\mathcal{S}_\nu$:

$$\omega_R \stackrel{\text{def}}{=} \Omega_D|_{\mathcal{S}_\nu} \equiv \Omega|_{\mathcal{S}_\nu}.$$

It is obvious that $\Omega_D|_{\mathcal{S}_\nu} \equiv \Omega|_{\mathcal{S}_\nu}$ since the last two terms in (26) vanish on $\mathcal{S}_\nu$.

Let us now assume that (Π, Ω) is a dual P–p pair in the sense of definitions 1 and 2. We will show that in the generic (Dirac) case and under certain conditions both pairs (Π_D, Ω_D) and (π_R, ω_R) are dual pairs.

Theorem 17. *Suppose that* (Π, Ω) *is a dual P–p pair with* $\ker(\Pi) = Sp\{dc_i\}_{i=1,\dots,r}$, $\ker(\Omega) = Sp\{Y_i\}_{i=1,\dots,r}$ *and with the corresponding foliation* $\mathcal{N}$ *of* $\mathcal{M}$. *Suppose also that the constraints* φ_j, $j = 1, \dots, k$, *define a foliation* $\mathcal{S}$ *of* $\mathcal{M}$ *with some transversal distribution spanned by the vector fields* Z_i *such that* Π *is* $\mathcal{Z}$*-invariant and that* $Z_i(\varphi_j) = \delta_{ij}$. *Suppose also that all* Y_i *are tangent to* $\mathcal{S}$ *(i.e.* $Y_i(\varphi_j) = 0$ *for all* i, j*) and that all* Z_i *are tangent to* $\mathcal{N}$ *(i.e.* $Z_i(c_j) = 0$ *for all* i, j*). Then the pair* (Π_D, Ω_D) *given by (18) and (26) has the following properties:*

1. $\ker(\Pi_D) = Sp\{\mathrm{d}c_i, \mathrm{d}\varphi_j\}$, $\ker(\Omega_D) = Sp\{Y_i, Z_j\}$.
2. *In the generic (Dirac) case when* Z_i *are obtained as in (21) the pair* (Π_D, Ω_D) *is a dual pair.*

Proof. We have already showed that $\Pi_D\,\mathrm{d}\varphi_i = 0$. A similar computation yields that $\Pi_D\,\mathrm{d}c_i = 0$. Using (26) we obtain

$$\begin{aligned}\Omega_D(Y_j) &= \Omega(Y_j) - \sum_{i=1}^{k} Y_j(\varphi_i)\xi_i + \sum_{i=1}^{k}\langle \xi_i, Y_j\rangle\,\mathrm{d}\varphi_i - \frac{1}{2}\sum_{i,l=1}^{k}\xi_i(Z_l)(Y_j(\varphi_l)\,\mathrm{d}\varphi_i - Y_j(\varphi_i)\,\mathrm{d}\varphi_i)\\ &= 0\end{aligned}$$

since $\Omega(Y_j) = 0$, $\langle \xi_i, Y_j\rangle = \Omega(Z_i, Y_j) = -\Omega(Y_j, Z_i) = 0$ and $Y_j(\varphi_i) = 0$ by assumption. Further

$$\Omega_D(Z_j) = \xi_j - \sum_{i=1}^{r} Z_j(\varphi_i)\xi_i + \sum_{i=1}^{k}\langle \xi_i, Z_j\rangle\,\mathrm{d}\varphi_i - \frac{1}{2}\sum_{i,l=1}^{k}\langle \xi_i, Z_l\rangle(\delta_{jl}\,\mathrm{d}\varphi_i - \delta_{ji}\,\mathrm{d}\varphi_l) = 0$$

all due to $Z_j(\varphi_i) = \delta_{ji}$. This concludes the proof of the first statement. Using some elementary tensor relations one can show (after some direct but cumbersome calculations) that the pair (Π_D, Ω_D) satisfies the following identity on $T\mathcal{M}$:

$$I = \Pi_D\Omega_D + \sum_{i=1}^{r} Y_i \otimes \mathrm{d}c_i + \sum_{j=1}^{k} Z_j \otimes \mathrm{d}\varphi_j - T \tag{27}$$

where the (1, 1)-tensor T is of the form

$$T = \sum_{i=1}^{k}\left(X_i + \sum_{j=1}^{k}\varphi_{ij} Z_i\right) \otimes \left(\xi_i + \sum_{l=1}^{k}\xi_l(Z_i)\,\mathrm{d}\varphi_l\right).$$

In the Dirac case the expressions in both parentheses in T vanish for every i due to (21) so that the whole tensor T vanishes. Further, the relations $Y_i(\varphi_j) = 0$, $Y_i(c_j) = \delta_{ij}$, $Z_i(\varphi_j) = \delta_{ij}$, $Z_i(c_j) = 0$ are just part of our assumptions. Thus, in the Dirac case all the requirements of definition 1 are satisfied. □

We are now in a position to prove the main theorem of this paper.

Theorem 18. *The pair* (π_R, ω_R) *obtained through the restriction* $(\pi_R, \omega_R) = (\Pi_D, \Omega_D)|_{\mathcal{S}_\nu}$ *of* (Π_D, Ω_D) *given by (18) and (26) is in the Dirac case a dual P–p pair on every leaf* $\mathcal{S}_\nu$ *of* $\mathcal{S}$ *with the Casimirs* $c_i|_{\mathcal{S}_\nu}$*, the kernel of* ω_R *spanned by* Y_i *(note that* Y_i *are tangent to* $\mathcal{S}_\nu$*) and with the partition of identity on* $T\mathcal{S}_\nu$ *given by*

$$I = \pi_R \omega_R + \sum_{i=1}^{r} Y_i \otimes \mathrm{d}\big(c_i|_{\mathcal{S}_\nu}\big). \tag{28}$$

Proof. The proof of this theorem follows from the fact that $(\pi_R, \omega_R) = (\Pi_D, \Omega_D)|_{\mathcal{S}_\nu}$. The partition of identity (28) follows from the partition (27) since $\mathrm{d}\varphi_i|_{\mathcal{S}_\nu} = 0$ and since $T = 0$ in the Dirac case. Further, π_R is Poisson since Π_D is. We only have to check that ω_R is closed.

Obviously, Ω_D is usually not closed, as according to (26) in the Dirac case we easily obtain

$$\mathrm{d}\Omega_D = \frac{1}{2}\sum_{i=1}^{k} \mathrm{d}\varphi_i \wedge \mathrm{d}\xi_i \neq 0.$$

However, $\omega_R = \Omega_D|_{\mathcal{S}_\nu}$ so that $\mathrm{d}\omega_R = \mathrm{d}(\Omega_D|_{\mathcal{S}_\nu}) = (\mathrm{d}\Omega_D)|_{\mathcal{S}_\nu} = 0$ due to the above formula, again since $\mathrm{d}\varphi_i|_{\mathcal{S}_\nu} = 0$. □

Thus, starting from a dual P–p pair (Π, Ω) and a proper foliation $\mathcal{S}$ (defined by a Dirac second-class constraint φ_i) we have constructed a dual P–p pair (π_R, ω_R) on every leaf of $\mathcal{S}$.

4. Examples

Let us now continue with examples 10 and 11. To illustrate our approach we will also construct the deformations Ω_D dual to the respective bivectors Π_D, although it is not necessary for the actual construction of the reductions ω_R that can be obtained directly by restricting Ω to $\mathcal{S}_\nu$.

Example 19 (Example 10 continued)**.** Assume that $n = 3$ and $r = 1$ so that the manifold $\mathcal{M}$ is of dimension 7 and the local coordinates are $(q_1, q_2, q_3, p_1, p_2, p_3, c)$. The original dual P–p pair (Π, Ω) is given by (12). Let us now introduce a five-dimensional submanifold $\mathcal{S}_0$ through the following pair of constraints:

$$\varphi_1 \equiv q_1 q_2 + q_3 = 0 \qquad \varphi_2 \equiv p_1 + p_2 q_1 + p_3 q_2 = 0 \tag{29}$$

(for some motivation on the source of these constraints, see [9]) so that $k = 2$ here. The constraints (29) do not contain the Casimir function c explicitly so that the condition $Y(\varphi_i) = 0$ is satisfied as $Y = \frac{\partial}{\partial c}$. The 2×2 matrix φ has the form

$$\varphi = \big(2q_2 + q_1^2\big)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The vector fields X_i and then Z_i can be easily computed (21). The result is

$$X_1 = -q_2\frac{\partial}{\partial p_1} - q_1\frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_3} \qquad X_2 = \frac{\partial}{\partial q_1} + q_1\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial q3} - p_2\frac{\partial}{\partial p_1} - p_3\frac{\partial}{\partial p_2}$$

$$Z_1 = \frac{1}{\varphi_{12}}X_2 \qquad Z_2 = -\frac{1}{\varphi_{12}}X_1$$

and one can see that $Z_i(c) = 0$ as Z_i do not contain derivations with respect to the coordinate variable c. A direct computation of expression (18) leads to

$$\Pi_D = \frac{1}{\varphi_{12}}\left[\begin{array}{cc|c} 0_3 & A & \multirow{2}{*}{$0_{6\times 1}$} \\ -A^t & B & \\ \hline \multicolumn{2}{c|}{0_{1\times 6}} & 0 \end{array}\right]$$

with

$$A = \begin{bmatrix} q_2 + q_1^2 & -q_1 & -1 \\ -q_1 q_2 & 2q_2 & -q_1 \\ -q_2^2 & -q_1 q_2 & q_2 + q_1^2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & p_2 q_1 - p_3 q_2 & p_2 \\ p_3 q_2 - p_2 q_1 & 0 & p_3 \\ -p_2 & -p_3 & 0 \end{bmatrix}.$$

It can be easily shown that Π_D is indeed Poisson. The 1-forms ξ_i are given by

$$\xi_1 = \frac{1}{\varphi_{12}}\, \mathrm{d}\varphi_2 \qquad \xi_2 = -\frac{1}{\varphi_{12}}\, \mathrm{d}\varphi_1$$

and (26) yields

$$\Omega_D = \frac{1}{\varphi_{12}} \left[\begin{array}{cc|c} -B & A^t & \multirow{2}{*}{$0_{6\times 1}$} \\ -A & 0 & \\ \hline \multicolumn{2}{c|}{0_{1\times 6}} & 0 \end{array}\right].$$

One can easily check that $\mathrm{d}\Omega_D \neq 0$. However, (Π_D, Ω_D) is a dual pair with $Y = \frac{\partial}{\partial c}$. In order to obtain explicit expressions for π_R and ω_R we pass to a new coordinate system $(q_1, q_2, \varphi_1, \varphi_2, p_2, p_3, c)$ as the constraints (29) are in a natural way solvable with respect to q_3, p_1. In these Casimir variables our operators attain the form

$$\Pi_D' = \frac{1}{\varphi_{12}} \left[\begin{array}{cc|c} 0_3 & A' & \multirow{2}{*}{$0_{6\times 1}$} \\ -A'^t & B' & \\ \hline \multicolumn{2}{c|}{0_{1\times 6}} & 0 \end{array}\right]$$

with

$$A' = \begin{bmatrix} 0 & -q_1 & -1 \\ 0 & 2q_2 & -q_1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & p_3 \\ 0 & -p_3 & 0 \end{bmatrix}$$

and

$$\Omega_D' = \left[\begin{array}{cc|c} C' & D' & \multirow{2}{*}{$0_{6\times 1}$} \\ -D'^t & 0_3 & \\ \hline \multicolumn{2}{c|}{0_{1\times 6}} & 0 \end{array}\right]$$

with

$$C' = \begin{bmatrix} 0 & p_3 & 0 \\ -p_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad D' = \begin{bmatrix} -1 & q_1 & 2q_2 \\ 0 & -1 & q_1 \\ \frac{1}{\varphi_{12}} & 0 & -1 \end{bmatrix}.$$

Now, if we parametrize the submanifold $\mathcal{S}_0$ with the coordinates (q_1, q_2, p_2, p_3, c) then we can immediately obtain the expressions for π_R and ω_R:

$$\pi_R = \frac{1}{\varphi_{12}} \begin{bmatrix} 0 & 0 & -q_1 & -1 & 0 \\ 0 & 0 & 2q_2 & -q_1 & 0 \\ q_1 & -2q_2 & 0 & p_3 & 0 \\ 1 & q_1 & -p_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \omega_R = \begin{bmatrix} 0 & p_3 & q_1 & 2q_2 & 0 \\ -p_3 & 0 & -1 & q_1 & 0 \\ -q_1 & -1 & 0 & 0 & 0 \\ -2q_2 & q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and it can be checked directly that it is indeed a dual P–p pair. For example, one can immediately check that

$$I = \pi_R \omega_R + Y \otimes \mathrm{d}\big(c\big|_{\mathcal{S}_0}\big).$$

The example presented is very simple, but illustrative. Let us now turn to our non-canonical example.

Example 20 (Example 11 continued)**.** This time we work with the dual P–p pair (13)–(14) written in coordinates (q_1, q_2, p_1, p_2, e). Observe that now e is not any Casimir variable (and hence a different letter to denote this odd variable). Let us now introduce a three-dimensional submanifold $\mathcal{S}_0$ through the following pair of constraints:

$$\varphi_1 \equiv p_1 - 1 = 0 \qquad \varphi_2 \equiv -p_2 p_1^2 + e q_1 p_1 + 2\ln\left(q_1^2\right) = 0 \tag{30}$$

so that $k = 2$ again. It is easy to check that $Y(\varphi_i) = 0$ and only a bit more difficult to see that $Z_i(c) = 0$ so that the assumptions of theorem 17 are again satisfied. Calculations similar to those in example 19 lead to

$$\Pi_D = \begin{bmatrix} 0 & \frac{q_1^2}{2p_1} & 0 & q_1 & p_1 \\ -\frac{q_1^2}{2p_1} & 0 & 0 & A & p_2 + \frac{4}{p_1^2} \\ 0 & 0 & 0 & 0 & 0 \\ -q_1 & -A & 0 & 0 & e + \frac{4}{q_1 p_1} \\ -p_1 & -p_2 - \frac{4}{p_1^2} & 0 & -e - \frac{4}{q_1 p_1} & 0 \end{bmatrix}$$

(p_1 is now a Casimir for Π_D) with $A = \frac{q_1 p_2}{p_1} + \frac{2q_1}{p_1^3} - \frac{e q_1^2}{2p_1^2}$ and to

$$\Omega_D = \begin{bmatrix} 0 & 2\frac{p_1}{q_1^2} & B & -2\frac{1}{q_1} & \frac{1}{2}p_1 p_2 - \frac{1}{2}e q_1 \\ -2\frac{p_1}{q_1^2} & 0 & -2\frac{1}{q_1} & 0 & -\frac{1}{2}p_1^2 \\ -B & 2\frac{1}{q_1} & 0 & 0 & C \\ 2\frac{1}{q_1} & 0 & 0 & 0 & \frac{1}{2}q_1 p_1 \\ \frac{1}{2}e q_1 - \frac{1}{2}p_1 p_2 & \frac{1}{2}p_1^2 & -C & -\frac{1}{2}q_1 p_1 & 0 \end{bmatrix}$$

with $B = 4\frac{q_2}{q_1^2} - 2\frac{e}{p_1^2} - 8\frac{q_1}{p_1^3}$, $C = p_1 q_2 - \frac{1}{2}q_1 p_2 - 2\frac{q_1}{p_1^2}$. It turns out that in this case Ω_D is closed so that here our pair (Π_D, Ω_D) is a dual P–p pair. In order to reduce this pair onto $\mathcal{S}$ we pass to the Casimir variables $(q_1, q_2, \varphi_1, \varphi_2, c)$ defined by constraints (30) together with (15),

$$\varphi_1 = p_1 - 1 \qquad \varphi_2 = -p_2 p_1^2 + e q_1 p_1 + 2\ln\left(q_1^2\right) \qquad c = -\tfrac{1}{2}q_2 p_1^2 + \tfrac{1}{2}q_1 p_1 p_2 - \tfrac{1}{4}e q_1^2.$$

It is possible to solve these equations with respect to p_1, p_2, e. We get the expressions

$$p_1 = p_1(q, \varphi, c) = \varphi_1 + 1 \qquad p_2 = p_2(q, \varphi, c) \qquad e = e(q, \varphi, c) \tag{31}$$

which are, however, too complicated to present explicitly. In these new variables the operator Π_D attains an almost canonical form

$$\Pi'_D = \frac{q_1^2}{2(\varphi_1 + 1)} \left[\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & 0_{2\times 3} \\ \hline \multicolumn{2}{c|}{0_{3\times 2}} & 0_{3\times 3} \end{array}\right]$$

while Ω_D attains a rather complicated form

$$\Omega'_D = \begin{bmatrix} 0 & -2\frac{p_1}{q_1^2} & 2\frac{e}{p_1^2} - 4\frac{q_2}{q_1^2} & -\frac{2}{p_1^2 q_1} & \frac{2p_1 p_2}{q_1^2} \\ 2\frac{p_1}{q_1^2} & 0 & -\frac{2}{q_1} & 0 & -2\frac{p_1^2}{q_1^2} \\ 4\frac{q_2}{q_1^2} - 2\frac{e}{p_1^2} & \frac{2}{q_1} & 0 & \frac{2}{p_1^3} & A' \\ \frac{2}{p_1^2 q_1} & 0 & -\frac{2}{p_1^3} & 0 & -\frac{2}{q_1 p_1} \\ -\frac{2p_1 p_2}{q_1^2} & 2\frac{p_1^2}{q_1^2} & -A' & \frac{2}{q_1 p_1} & 0 \end{bmatrix}$$

with $A' = -2\frac{p_2}{q_1} + 2\frac{e}{p_1} - 4\frac{p_1 q_2}{q_1^2}$ and with p_1, p_2, e given by (31). We are now ready to reduce our operators onto $\mathcal{S}_0$. The result is

$$\pi_R = \frac{q_1^2}{2}\left[\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & 0_{2\times 1} \\ \hline \multicolumn{2}{c|}{0_{1\times 2}} & 0 \end{array}\right] \qquad \omega_R = \frac{2}{q_1^2}\begin{bmatrix} 0 & -1 & B \\ 1 & 0 & -1 \\ B & 1 & 0 \end{bmatrix}$$

with $B = \left(4c + 2q_2 - 2q_1 \ln\left(q_1^2\right)\right)/q_1$. Again it can be checked directly that it is indeed a dual P–p pair.

5. Conclusions

In this paper we have constructed a theory of Dirac-type reductions for Poisson bivectors and presymplectic (i.e. closed but in general degenerate) 2-forms by embedding them in a geometrical object that we call a 'dual pair'. We systematically constructed the theory of dual pairs and of their special type: Poisson–presymplectic pairs (i.e. dual pairs consisting of one Poisson operator and one closed 2-form). Using this theory we presented how to project in principle any dual P–p pair onto submanifolds in such a way that the reduced pair is again a dual P–p pair. Our method is in a sense a generalization of the concepts of Dirac, Marsden and Ratiu. We concluded the paper by two examples: one starting from a canonical dual pair and one non-canonical.

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